Triangle of Velocities
and
Mathematical Invalidity of the Lorentz Transformation
in Special Relativity

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Abstract

Lorentz Transformation as interpreted within the context of Special relativity is mathematically incorrect set of equations. This article presents the general case proof of invalidity, independent of derivation procedure.

Author presents new solution named Triangle of Velocities, which is logically and mathematically correct interpretation of the Lorentz transformation.

1. Triangle of Velocities

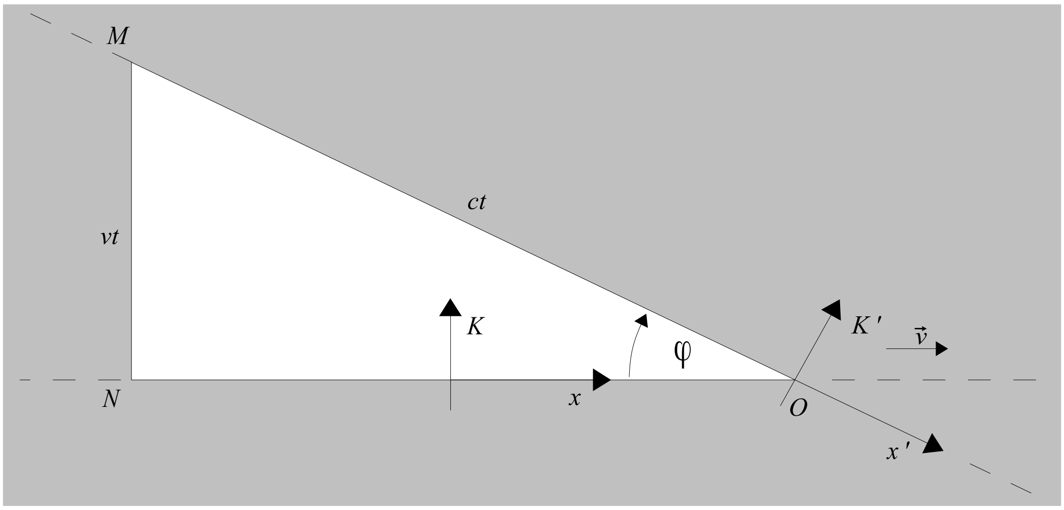
The linear equation set which is usually associated with transformation of coordinates,

$$\begin{aligned} x' &= Ax + Bt \\ t' &= Cx + Dt \end{aligned} \qquad (1.1)$$

has very simple mathematical solution, and we will derive it now. The key to solution is that these equations do not contain information on how are x and x' axes oriented relative to one another.

The triangle in picture below is formed by two coordinate systems K and K' which travel along straight line at constant speed v relative to one another. Time is reset in both systems as they pass by one another ($t=0$, $t'=0$ when $x=0$, $x'=0$).

To define the triangle we use the following setup: length MN is equal to the distance traveled by K', and hypotenuse MO is distance traveled by a ray of light at speed c in the same amount of time t. The idea is to use speeds v and c to define the angle between the two systems.



Angle ϕ is defined then with $\sin\phi = \frac{vt}{ct} = \frac{v}{c}$. Because of trigonometrical identity $\sin^2\phi + \cos^2\phi = 1$ we have

$$\cos\phi = \sqrt{1 - \frac{v^2}{c^2}} \qquad (1.2)$$

Equation (1.2) will play key role in transformation of coordinates (1.1) as it tells us how lengths in K' relate to those in K . First equation from 1.1. can be written as

$$x' = A\left(x - \left(-\frac{B}{A}\right)t\right) \qquad (1.3)$$

For all events at origin of K' we have $x'=0$ and $x=vt$. By substituting this in (1.3) we find that speed of K' relative to K is $v = -\frac{B}{A}$, and (1.3) becomes:

$$x' = A(x - vt) \tag{1.4}$$

There is symmetry in that speed of arbitrary material point in both systems must be the same, but of opposite sign: $NO/t = -OM/t'$. Writing this in differential form we have:

$$v = \frac{dx}{dt} = -\frac{B}{A} = -\frac{dx'}{dt'} = -\frac{d(Ax+Bt)}{d(Cx+Dt)} \tag{1.5}$$

For all events at origin of K, we have $x=0$, so (1.5) becomes

$$\frac{B}{A} = \frac{B}{D} \text{ or } A = D \tag{1.6}$$

Next, we can write second equation from (1.1) in the following form:

$$t' = A(t + Ex) \tag{1.7}$$

where $E = C/A$. To find out E, we will create a class of events that involve rays of light. We demand that transformation must be applicable to events for which the following is true:

$$x = ct \tag{1.8}$$
$$x' = ct' \tag{1.9}$$

These are valid events in triangle, as is demonstrated in examples following this derivation.

From (1.4), (1.7), (1.8) and (1.9) we find $E = -\frac{v}{c^2}$. Now our transformation becomes:

$$\begin{aligned} x' &= A(x - vt) \\ t' &= A\left(t - \frac{v}{c^2}x\right) \end{aligned} \tag{1.11}$$

Inverse of (1.11) is:

$$\begin{aligned} x &= A(x' + vt') \\ t &= A\left(t' + \frac{v}{c^2}x'\right) \end{aligned} \tag{1.12}$$

After substituting (1.12) into (1.11) we find $A = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$. Note that we could have skipped using (1.11) and (1.12) as this value of A is immediately obvious from the triangle and equations (1.2) and (1.4) since $x' = (x - vt)/\cos\phi$.

Finally we have solution known as “Lorentz transformation”:

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}},\ t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{1.13}$$

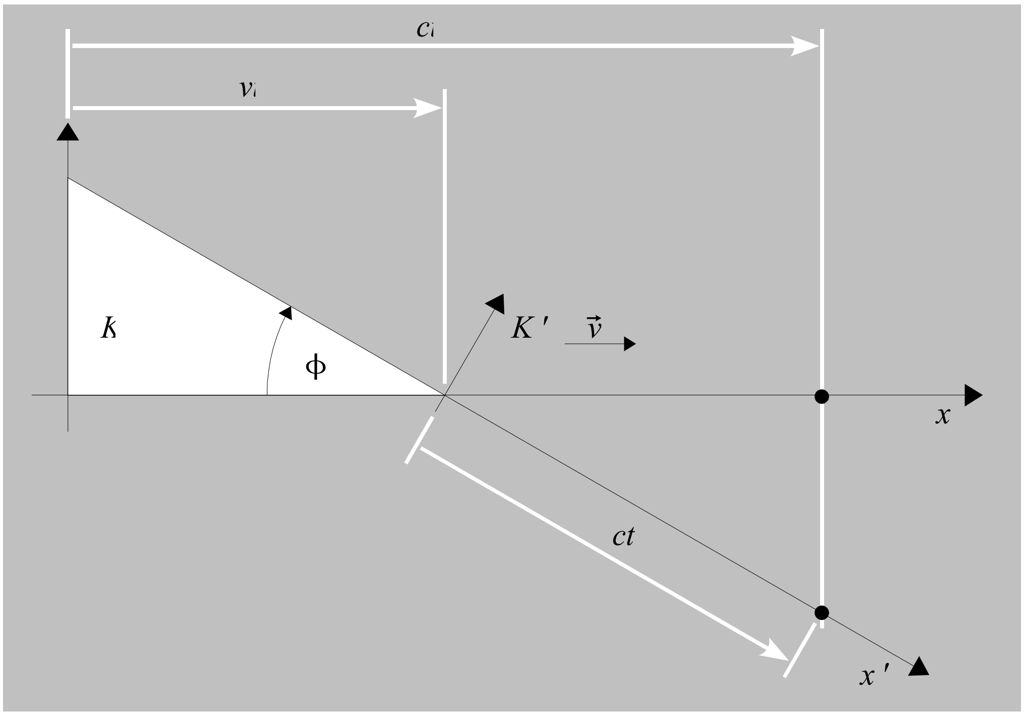
1.1. Examples

Example 1. Triangle of velocities is defined with $v=0.866c$. What is the angle between adjacent leg and hypotenuse?

We have seen in (1.2) that cosine of that angle is $\cos\phi=\sqrt{1-\frac{v^2}{c^2}}=0.5$

Therefore $\phi=\arccos 0.5=60^0$

Example 2. We can illustrate the case for which both $x=ct$ and $x'=ct'$ hold true. These lengths are marked on the vertical line to the right.



This figure is properly scaled. To create it, the following was used: by substituting (1.9) into (1.5) and (1.9) into (1.12) we find

$$x'=\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}\,x\,,\quad t'=\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}\,t$$

With $\phi=30^0$, using (1.2) we find $\frac{v}{c}=0.5$, so above becomes $x'=0.577x$, $t'=0.577t$. The dimensions are: $x=ct=100\,mm$, $vt=50\,mm$ and $x'=ct'=57.7\,mm$.

Different values for time t and t' simply mean that in K' ray of light has smaller distance to travel until condition $x'=ct'$ is satisfied, opposed to longer distance for which $x=ct$ holds true. So for $x'<x$, naturally we have $t'<t$. Without knowing Lorentz transformation, these same results can be obtained from the relations which are obvious from the figure:

$$vt+ct'\cos\phi=ct \text{ or numerically: } 50+57.7\cdot\cos 30^0=100 .$$

2. Mathematical Invalidity of Lorentz Transformation in Special Relativity

2.1 Derivation of the Lorentz transformation

The Lorentz transformation in Special relativity is derived for two parallel coordinate systems K and K' in relative uniform motion, with clocks reset to zero as they pass by one another. We start with assumption that transformation of coordinates must be linear:

$$\begin{aligned} x' &= Ax + Bt \\ t' &= Cx + Dt \end{aligned} \qquad (2.1)$$

Another assumption which is rarely stated explicitly is that this transformation is expected to work for arbitrary events, completely unrelated to motion of coordinate systems themselves. An event is any pair (x, t), and goal is to find appropriate (x', t').

First equation from (2.1) can be written as:

$$x' = A\left(x - \left(-\frac{B}{A}\right)t\right) \qquad (2.2)$$

For all events at origin of K' we have $x' = 0$ and $x = vt$. By substituting this in (2.2) we find that speed of K' relative to K is $v = -\frac{B}{A}$, and (2.2) becomes:

$$x' = A(x - vt) \qquad (2.3)$$

There is a symmetry in that speed v of K' relative to K must be equal to speed of K relative to K', but of opposite sign. Writing this in differential form we have:

$$v = -\frac{B}{A} = -\frac{dx'}{dt'} = -\frac{d(Ax + Bt)}{d(Cx + Dt)} \qquad (2.4)$$

For all events at origin of K, we have $x = 0$, so (2.4) becomes:

$$\frac{B}{A} = \frac{B}{D} \quad \text{or} \quad A = D \qquad (2.5)$$

Next, we can write second equation from (2.1) in the following form:

$$t' = A(t + Ex) \qquad (2.6)$$

where $E = C/A$. The transformation must be valid for all events traveling at the speed of light relative to origin of K':

$$x = (c + v)t \qquad (2.7)$$

$$x' = ct' \qquad (2.8)$$

Special relativity states the following for the speed of light:

$$c + v = c \qquad (2.9)$$

By substituting (2.9) in (2.7) and using it with (2.3), (2.6) and (2.8) we find $E=-\frac{v}{c^2}$.

Now our transformation becomes:

$$\begin{aligned} x' &= A(x-vt) \\ t' &= A\left(t-\frac{v}{c^2}x\right) \end{aligned} \qquad (2.10)$$

Inverse of (2.10) is:

$$\begin{aligned} x &= A\,(x'+v\,t') \\ t &= A\left(t'+\frac{v}{c^2}x'\right) \end{aligned} \qquad (2.11)$$

After substituting (2.11) into (2.10) we find $A=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

Finally we have solution

$$x'=\frac{x-vt}{\sqrt{1-\frac{v^2}{c^2}}}\,,\; t'=\frac{t-\frac{v}{c^2}x}{\sqrt{1-\frac{v^2}{c^2}}} \qquad (2.12)$$

There are also some other derivation procedures, but they happen to be variations in style only, not in essence, as can be seen in various literature and on the Internet.

2.2. Explanation of Errors

In the derivation of Lorentz transformation in Special relativity, serious logical errors were made. Those are introduction of “invariant” number, use of weak induction to assume that transformation is applicable to arbitrary events, and making assumption that procedure is applicable to parallel systems, when procedure does not contain any information to support that claim.

We will examine these errors in detail.

2.2.1. “Invariant” number

As speed v in the transformation is different from zero, equation (2.9) means that c is a very special number for which arithmetic doesn't work. This of course is a nonsense, equivalent of saying 2 + 1 = 2, for very special number 2.

(2.9) is required as without it the procedure would resolve into $x'=x-vt\,,\; t'=t$ (which is Galilean transformation for parallel systems).
To avoid saying that for the number c arithmetic doesn't work, it was given a special name – “invariant” number. This “reasoning” is specific of Special relativity only, since mathematics does not know of “invariant” numbers.

The key difference of the derivation in physics textbooks is writing (2.7) and (2.8) together as

$$x=ct$$
$$x'=ct'$$

where the key equation (2.9) is implicit , and c is immediately called “invariant”, which masks the error.

2.2.2. Inapplicability to arbitrary events

It is very important to understand that Lorentz transformation is inapplicable to arbitrary events. Equations (2.12) were derived **only for the following classes of events**:

$$x=vt \text{ with } x'=0\,,$$
$$x=0 \text{ with } x'=-vt'$$
$$x=ct \text{ with } x'=ct'$$

All three conditions are equations of motion, and time in all of them is time required for certain distance to be traveled.

Related logical error of Special relativity was making conclusion silently through **confirmation bias**: a derivation for arbitrary events was **desired**, such that for any (x,t) we can find appropriate (x',t'), and after using procedure identical with triangle of velocities, obtained transformation was silently assumed to work for any events.

In triangle of velocities, arbitrary events (e.g. a light bulb being turned on at $x=1, t=0$) are meaningless as they conflict with meaning of time in above conditions, **where time is always associated with motion**, and therefore cannot be zero for nonzero traveled length ($x=vt=0$ or $x=ct=0$, therefore $x\neq 1$ when $t=0$).

2.2.3. Absence of information that K and K' are parallel

As the procedure starts, it is said that K and K' are **parallel to one another**, and linear equation set (2.1) is laid out as starting point of the procedure.

Prior to (2.7), procedure does not carry information on angle between the two systems (it can be any).

We stated in (2.7) that speed of ray of light relative to K is $c+v$. This is the only place in procedure holding information that the two systems are parallel.

As (2.9) is used in (2.7) it becomes $x=ct$. By replacing $x=(c+v)t$ with $x=ct$ information that the two systems are parallel **is discontinued**, while at the same time making entire procedure identical to the one used for triangle of velocities.